

**Non-modal kinetic theory of the hydrodynamic drift instabilities of plasma shear flows**V. V. Mikhailenko,<sup>1</sup> V. S. Mikhailenko,<sup>2</sup> and Hae June Lee<sup>1</sup><sup>1</sup>*Department of Electrical Engineering, Pusan National University, Busan 609–735, S. Korea.*<sup>a)</sup><sup>2</sup>*Department of Physics and Technology, V.N. Karazin Kharkov National University, 61108 Kharkov, Ukraine.*

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The non-modal kinetic theory of the kinetic drift instability of plasma shear flows [Phys.Plasmas, 18, 062103 (2011)] is extended to the investigation of the long-time evolution of the hydrodynamic ion temperature gradient and resistive drift instabilities in plasma shear flow. We find, that these hydrodynamic instabilities passed in their temporal evolution in shear flow through the kinetic stage of the evolution. In linear theory, this evolution involves the time dependent, due to flow shear, effects of the finite Larmor radius, which resulted in the non-modal effect of the decrease with time the frequencies and the growth rates of the instabilities.

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## I. INTRODUCTION

The investigation of the long-time mode of behaviour of the regimes of tokamak plasma improved confinement, where  $E \times B$  flow shear stabilization mechanisms<sup>1</sup> plays a key role, is the theme of major importance in the physics of continuous tokamaks operation. Gyrokinetic theory<sup>2</sup> gives a foundation for the investigating microinstabilities, which cause the anomalous transport in fusion plasmas. While there has been significant progress in gyrokinetic treatment of tokamak turbulence (see, for example, Refs.<sup>3-12</sup>), there key issues still remain in application of the gyrokinetic theory to analytical investigations of the long-time evolution of plasma shear flows turbulence.

In Ref.<sup>13</sup> we have developed new kinetic description of the plasma shear flows, using the Kelvin's method of shearing modes or so-called non-modal approach as its foundation. The governing equation in that theory is the integral equation for the perturbed electrostatic potential, which is the formal solution of the initial value problem for Vlasov-Poisson (V-P) system. The important procedure in its derivation is the transformation of V-P system to sheared (in space and velocity) coordinates convected with flow and accounting for by this mean the effect of spatial time-dependent distortion of plasma disturbances by shear flows. In that equation, velocity shear reveals as the non-modal time-dependent effect of the finite Larmor radius. Because of the shearing of perturbations in shear flow, the component of the wave number along the direction of the velocity shear experiences the growth with time, and therefore results obtained for the hydrodynamic drift instabilities on the base of the fluid description are valid only for finite times till  $k_{\perp}\rho_i \ll 1$  is valid ( $\rho_i$  is thermal ion Larmor radius). Therefore, not only kinetic, but also typically hydrodynamic instabilities of shearless plasma require the kinetic description for plasma shear flows for proper treating their evolution on times, at which  $k_{\perp}\rho_i$  becomes not small and approaches unity and above. It reveals<sup>13</sup>, that conventional kinetic theory, as well as the gyrokinetic theory, in which the time dependence of the perturbed distribution functions and fields are considered in a canonical modal form,  $\varphi \sim \exp(-i\omega t)$ , are valid only at the initial stage of the shear flow turbulence evolution, at times less then the inverse velocity shearing rate  $(V_0')^{-1}$ . The theory developed displays that the experimentally observed suppression of drift-type turbulence and the improved energy confinement are appreciable at  $t \gtrsim (V_0')^{-1} \gtrsim (\gamma)^{-1}$ , i.e. just after that time. In our analysis of the kinetic drift instability we have obtained<sup>13</sup>, that in linear theory

of the kinetic (universal) drift instability, shear flow leads at time  $t \gtrsim (V_0')^{-1}$  to the non-modal decrease with time the frequency and growth rate and to ultimate suppression of that instability.

The basic transformation of the V-P system to sheared coordinates, which resulted in the removing from the Vlasov equation the term, which contains the spatial inhomogeneity introduced by the velocity shear, is presented and discussed in Sec. II. In Sec. III, we apply the linear non-modal kinetic theory of Ref.<sup>13</sup>, extended onto the accounting for the inhomogeneity of the ion temperature, to the investigation of the temporal evolution of the hydrodynamic ion temperature gradient drift instability of plasma shear flow. We find that the ordinary modal theory of this instability is valid only for times less than the inverse velocity shearing rate. At times  $t_s \gg t \gtrsim (V_0')^{-1}$ , where  $t_s = (V_0' k_y \rho_i)^{-1}$ , the non-modal effects of the decrease of the frequency and growth rate develop. At the final stage, at times  $t \gg t_s$  the perturbations of the cell type with zero frequency occurs.

In Sec.IV, using the integral equation for electrostatic potential obtained in Ref.<sup>13</sup>, extended onto the accounting for the collision of electrons with neutrals, we consider temporal evolution of the resistive drift instability of plasma with comparable ion and electron temperatures, as it is at the edge layer of tokamaks. We came to the same conclusion about the importance of the kinetic theory for the proper treating of the resistive drift instability in shear flow. It is shown, that the reducing with time the frequency and growth rate is a common future of the evolution of the electrostatic drift instabilities in plasma shear flows. A summary of the work is given in Conclusions, Section V.

## II. VLASOV–POISSON SYSTEM OF EQUATIONS IN SHEARED COORDINATES

We start with the Vlasov equation for species  $\alpha$  ( $\alpha = i$  for ions and  $\alpha = e$  for electrons), immersed in crossed spatially inhomogeneous electric field,  $\mathbf{E}_0(\hat{\mathbf{r}})$  and homogeneous magnetic field  $\mathbf{B} \parallel \mathbf{e}_z$ ,

$$\frac{\partial F_\alpha}{\partial t} + \hat{\mathbf{v}} \frac{\partial F_\alpha}{\partial \hat{\mathbf{r}}} + \frac{e}{m_\alpha} \left( \mathbf{E}_0(\hat{\mathbf{r}}) + \frac{1}{c} [\hat{\mathbf{v}} \times \mathbf{B}] - \nabla \varphi(\hat{\mathbf{r}}, t) \right) \frac{\partial F_\alpha}{\partial \hat{\mathbf{v}}} = 0. \quad (1)$$

We use a slab geometry with the mapping  $(r, \theta, \varphi) \rightarrow (\hat{x}, \hat{y}, \hat{z})$  where  $r, \theta, \varphi$  are the radial, poloidal and toroidal directions, respectively, of the toroidal coordinate system. In this

paper, we consider the case of plasma shear flow in linearly changing electric field,  $\mathbf{E}_0(\hat{\mathbf{r}}) = (\partial E_0/\partial \hat{x}) \hat{x} \mathbf{e}_x$  with  $\partial E_0/\partial \hat{x} = \text{const}$ . In that case

$$\mathbf{V}_0(\mathbf{r}) = V_0(\hat{x}) \mathbf{e}_y = -\frac{c}{B} \frac{\partial E_0}{\partial \hat{x}} \hat{x} \mathbf{e}_y = V'_0 \hat{x} \mathbf{e}_y. \quad (2)$$

with spatially homogeneous,  $V'_0 = \text{const}$ , velocity shear. The possible spatially homogeneous part of shear flow velocity is eliminated from the problem by a simple Galilean transformation. It was obtained in Ref.<sup>13</sup>, that transition in the Vlasov equation from velocity  $\hat{\mathbf{v}}$  and coordinates  $\hat{x}, \hat{y}, \hat{z}$  to convected coordinates  $\mathbf{v}$  in velocity space, determined by

$$\hat{v}_x = v_x, \quad \hat{v}_y = v_y + V'_0 x, \quad \hat{v}_z = v_z \quad (3)$$

and to sheared with flow coordinates  $x, y, z$  in the configurational space, determined by

$$\hat{x} = x, \quad \hat{y} = y + V'_0 t x, \quad \hat{z} = z \quad (4)$$

(it is assumed that inhomogeneous electric field, and respectively shear flow originate at time  $t = t_{(0)} = 0$ ) transforms the linearized Vlasov equation for the perturbation of the distribution function  $f_\alpha = F_\alpha - F_{0\alpha}$ , with known equilibrium distribution  $F_{0\alpha}$ , to the form,

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} + v_{\alpha x} \frac{\partial f_\alpha}{\partial x} + (v_{\alpha y} - v_{\alpha x} V'_0 t) \frac{\partial f_\alpha}{\partial y} + v_{\alpha z} \frac{\partial f_\alpha}{\partial z} + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - (\omega_{c\alpha} + V'_0) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} \\ = \frac{e_\alpha}{m_\alpha} \left( \frac{\partial \varphi}{\partial x} - V'_0 t \frac{\partial \varphi}{\partial y} \right) \frac{\partial F_{0\alpha}}{\partial v_{\alpha x}} + \frac{e_\alpha}{m_\alpha} \frac{\partial \varphi}{\partial y} \frac{\partial F_{0\alpha}}{\partial v_{\alpha y}} + \frac{e_\alpha}{m_\alpha} \frac{\partial \varphi}{\partial z} \frac{\partial F_{0\alpha}}{\partial v_{\alpha z}}. \end{aligned} \quad (5)$$

( $\omega_c$  is the cyclotron frequency of ion (electron)) which is free from the spatial inhomogeneities originated from shear flow (see also Eq.(8) in Ref.<sup>13</sup>). The Fourier transformation of Eq.(5) over spatial coordinates  $x, y, z$  with the electrostatic potential  $\varphi(\mathbf{r}, t)$  determined as a function of coordinates  $x, y, z$  as

$$\varphi(x, y, z, t) = \int \varphi(k_x, k_y, k_z, t) e^{ik_x x + ik_y y + ik_z z} dk_x dk_y dk_z, \quad (6)$$

gives the equation for the separate spatial Fourier harmonic of  $f_\alpha$  with wave numbers  $k_x, k_y, k_z$

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} + (i(k_x - V'_0 t k_y) v_{\alpha x} + i k_y v_{\alpha y} + i k_z v_{\alpha z}) f_\alpha(\mathbf{v}_\alpha, \mathbf{k}, t) + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - (\omega_{c\alpha} + V'_0) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} \\ = i \frac{e_\alpha}{m_\alpha} \varphi(\mathbf{k}, t) \left[ (k_x - V'_0 t k_y) \frac{\partial F_{0\alpha}}{\partial v_{\alpha x}} + i k_y \frac{\partial F_{0\alpha}}{\partial v_{\alpha y}} + i k_z \frac{\partial F_{0\alpha}}{\partial v_{\alpha z}} \right]. \end{aligned} \quad (7)$$

in which wave numbers  $k_x, k_y, k_z$  and time change independently. It follows from Eqs.(5) and (7), that the transformation of the Vlasov equation to convected-sheared coordinates

(3), (4) converts the spatial inhomogeneity into the time inhomogeneity. That prevents the application of the spectral transforms in time to Eq.(7) and to obtain the ordinary dispersion equations valid for any times. It reveals<sup>13</sup> that the ordinary modal solution to Eq.(7), for which ordinary dispersion equation may be obtained, exists only at time  $t \ll (V'_0)^{-1}$ ; solution becomes non-modal<sup>13</sup> with time dependent frequency and growth rate for larger time. With coordinates  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  of the laboratory set of references, transformation (6) has a form

$$\begin{aligned}\varphi(\hat{x}, \hat{y}, \hat{z}, t) &= \int \varphi(k_x, k_y, k_z, t) e^{ik_x \hat{x} + ik_y(\hat{y} - V'_0 t \hat{x}) + ik_z \hat{z}} dk_x dk_y dk_z \\ &= \int \varphi(k_x, k_y, k_z, t) e^{i(k_x - V'_0 t k_y) \hat{x} + ik_y \hat{y} + ik_z \hat{z}} dk_x dk_y dk_z.\end{aligned}\quad (8)$$

It follows from Eq.(8), that separate spatial Fourier mode in convected-sheared coordinates becomes a sheared mode with time dependent wave number  $k_x - V'_0 t k_y$  in the laboratory frame.

Usually, however, only the transformation to the convected coordinates (3) in velocity space, without the transformation to the sheared coordinates (4) in the configuration space, is used in the kinetic theory of plasma shear flows<sup>11,16</sup>. After such transformation the linearized Vlasov equation becomes

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t} + V'_0 \hat{x} \frac{\partial f_\alpha}{\partial \hat{y}} + \hat{\mathbf{v}} \frac{\partial f_\alpha}{\partial \hat{\mathbf{r}}} + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - (\omega_{c\alpha} + V'_\alpha) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} \\ = \frac{e_\alpha}{m_\alpha} \nabla \varphi(\hat{\mathbf{r}}, t) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}.\end{aligned}\quad (9)$$

Traditionally<sup>11,16</sup>, in proceeding with derivation of the governing equation for  $f_\alpha$ , the spatial Fourier transform in the laboratory configuration space,

$$\varphi(\hat{x}, \hat{y}, \hat{z}, t) = \int \varphi(\hat{k}_x, \hat{k}_y, \hat{k}_z, t) e^{i\hat{k}_x \hat{x} + i\hat{k}_y \hat{y} + i\hat{k}_z \hat{z}} d\hat{k}_x d\hat{k}_y d\hat{k}_z \quad (10)$$

for the electrostatic potential  $\varphi$  and for  $f_\alpha$  is adapted with assumption of the "slow variation" of  $V_0(x)$  with spatial coordinates. That gives the following equation for  $f_\alpha(t, \hat{\mathbf{k}}, \hat{\mathbf{v}})$ :

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t} - i\hat{k}_y V_0(\hat{x}) f_\alpha - i(\hat{\mathbf{k}} \hat{\mathbf{v}}) f_\alpha + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - (\omega_{c\alpha} + V'_\alpha) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} \\ = -\frac{e_\alpha}{m_\alpha} \hat{\mathbf{k}} \varphi(\hat{\mathbf{k}}, t) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}.\end{aligned}\quad (11)$$

In Eq.(10), the flow velocity shear reveals in the formation of elliptical orbits of particles, with velocity coordinates  $v_\perp$ ,  $\phi$ <sup>15</sup>

$$v_x = v_\perp \cos \phi, \quad v_y = \sqrt{\eta} v_\perp \sin \phi, \quad \phi = \phi_1 - \sqrt{\eta} \omega_c t, \quad v_z = v_z, \quad (12)$$

where  $\eta = 1 - V'_0/\omega_c$ , and with modified gyro-frequency  $\sqrt{\eta}\omega_c$ . That effect, however is negligible for  $|V'_0| \ll \omega_{ci}$ . Then, velocity shear is absorbed into the identical for both plasma species Doppler shifted frequency<sup>11,16</sup>  $\hat{\omega} = \omega - k_y V_0(\hat{x})$ , and, in fact, is excluded from the subsequent analysis.

Let us analyse the results of the application of the Fourier transformation (10) to Eq.(9) without the assumption of the "slow" spatial variation of the flow velocity. That gives the following differential equation in wave-number space:

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} - V'_0 \hat{k}_y \frac{\partial f_\alpha}{\partial \hat{k}_x} - i \left( \hat{\mathbf{k}} \hat{\mathbf{v}} \right) f_\alpha + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - (\omega_{c\alpha} + V'_\alpha) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} \\ = - \frac{e_\alpha}{m_\alpha} \hat{\mathbf{k}} \varphi \left( \hat{\mathbf{k}}, t \right) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}. \end{aligned} \quad (13)$$

It is interesting to know the relation between solutions of Eqs. (7) and (13) for  $f_\alpha$ . For the receiving from Eq.(13) the equation which couples  $f_\alpha$  and  $\varphi$  of the separate spatial Fourier mode, as it is in Eq.(5), we have to exclude from Eq.(13) the differential operator  $-V'_0 \hat{k}_y \frac{\partial f_\alpha}{\partial \hat{k}_x}$ , due to which the Fourier mode of  $f_\alpha$  appears to be coupled with all Fourier modes of the electrostatic potential and depends on the integral of  $\varphi$  over wave-number space. The characteristic equation

$$dt = - \frac{d\hat{k}_x}{V'_0 \hat{k}_y} \quad (14)$$

gives the solution  $\hat{k}_x + V'_0 t \hat{k}_y = K_x$ , where  $K_x$  as the integral of Eq.(13) is time independent. It reveals that  $f_\alpha = f_\alpha(K_x, \hat{k}_y, \hat{k}_z, t) = f_\alpha(\hat{k}_x + V'_0 t \hat{k}_y, \hat{k}_y, \hat{k}_z, t)$ , i.e. the wave number components  $\hat{k}_x$  and  $\hat{k}_y$  have to be changed in such a way that  $\hat{k}_x + V'_0 t \hat{k}_y$  leaves unchanged with time. The solution to Eq.(13) for  $f_\alpha$  can't be presented in the laboratory coordinates in a form, in which the time and spatial dependences are separable, as it is for the normal mode solutions of Eq. (11) obtained with assumption of the "slow" spatial variation of the flow velocity<sup>11</sup>. If we use, however,  $\hat{k}_x = K_x - V'_0 t \hat{k}_y$  in Eqs.(10) and (13), we obtain for the electrostatic potential the presentation (8), and we obtain Eq.(7) for  $f_\alpha$ , with time independent  $K_x = k_x$ ,  $\hat{k}_y = k_y$ ,  $\hat{k}_z = k_z$ . The obtained results prove, that the solution of the Vlasov equation in the form of the separate Fourier harmonic with time independent wave numbers may be obtained only in convected-sheared coordinates. That solution reveals in the laboratory frame as a shearing mode (8) with time dependent  $x$ -component of the wave number.

So, we have two procedures for the proper performing of the spatial Fourier transformation of the Vlasov equation for plasma shear flow, which give the same result. The first one is to apply at first the transformation (2) to convected-sheared coordinates, and then, to perform the Fourier transform of the Vlasov equation over spatial coordinates with time independent wave numbers  $k_x, k_y, k_z$ . The second procedure is to come to these time independent wave numbers through the solution of the characteristic equation Eq.(14), when the transformation to sheared coordinates in configuration space in the Vlasov equation does not perform. The sheared flow leads to the observed in the laboratory frame continuous inclining with time of the plane of constant phase the waves. That reveals in the observed in the laboratory frame the time dependence of the wave number  $\hat{k}_x = K_x - V'_0 \hat{k}_y t$  in the solution for the  $f_\alpha$ . The oversimplification of the problem, which resulted from the application of the assumption of "slow spatial variation of  $V_0(x)$ ", leads to the overlooking of that principal effect of shear flow. It is obvious, that the time dependence in  $K_x$  may be neglected only in the case of negligible velocity shear, or when the very short evolutionary time is considered. Really, for  $\hat{k}_y \sim \hat{k}_x$  and  $V'_0 \simeq \gamma$  for time  $t \gtrsim \gamma^{-1}$  the we have  $\hat{k}_y V'_0 t \gtrsim \hat{k}_x$  in the integral  $K_x = \hat{k}_x + V'_0 \hat{k}_y t$ . Therefore the assumption of "slow spatial variation of flow velocity" is not valid for the investigations of the effects of shear flow in real experiments, where observed velocity shearing rate may be of the order or above of the growth rate of the instability and the time of the observations is of the order of the inverse growth rate or longer.

With leading center coordinates  $X, Y$ , determined in convective-shearing coordinates by the relations

$$\begin{aligned} x &= X - \frac{v_\perp}{\sqrt{\eta}\omega_c} \sin \phi, & y &= Y + \frac{v_\perp}{\eta\omega_c} \cos \phi + V'_0 t (X - x), \\ z_1 &= z - v_z t, \end{aligned} \quad (15)$$

the Vlasov equation (1), in which species index is suppressed, transforms into the form<sup>13</sup>

$$\begin{aligned} &\frac{\partial F}{\partial t} + \frac{e}{m\sqrt{\eta}\omega_c} \left( \frac{\partial \varphi}{\partial X} \frac{\partial F}{\partial Y} - \frac{\partial \varphi}{\partial Y} \frac{\partial F}{\partial X} \right) \\ &+ \frac{e}{m} \frac{\sqrt{\eta}\omega_c}{v_\perp} \left( \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F}{\partial v_\perp} - \frac{\partial \varphi}{\partial v_\perp} \frac{\partial F}{\partial \phi_1} \right) - \frac{e}{m} \frac{\partial \varphi}{\partial z_1} \frac{\partial F}{\partial v_z} = 0, \end{aligned} \quad (16)$$

in which any time dependent coefficients are absent. It follows from Eqs.(15), that in shearing coordinates a particle gyro-motion is different from the ones in convective coordinates, determined by Eqs.(12). Now it consists in the rotation with modified gyro-frequency and stretching of gyro-orbit along coordinate  $y$  with velocity  $-V'_0(x - X)$ , which is negative for

$x > X$  and is positive for  $x < X$ . With leading center coordinates (15) the Fourier transform (6) becomes

$$\begin{aligned}
\varphi(x, y, z, t) &= \int \varphi(k_x, k_y, k_z, t) \exp \left[ ik_x \left( X - \frac{v_\perp}{\sqrt{\eta}\omega_c} \sin \phi \right) \right. \\
&\quad \left. + ik_y \left( Y + \frac{v_\perp}{\eta\omega_c} \cos \phi \right) + iV_0't \frac{k_y v_\perp}{\sqrt{\eta}\omega_c} \sin \phi + ik_z z \right] dk_x dk_y dk_z \\
&= \int \varphi(k_x, k_y, k_z, t) \exp [ik_x X + ik_y Y + ik_z z \\
&\quad + i \frac{k_y v_\perp \cos \phi}{\eta\omega_c} - i \frac{(k_x - V_0't k_y) v_\perp \cos \phi}{\sqrt{\eta}\omega_c}] dk_x dk_y dk_z = \\
&= \int \varphi(k_x, k_y, k_z, t) \exp [ik_x X_i + ik_y Y_i + ik_z z \\
&\quad - i \frac{\hat{k}_\perp(t) v_\perp}{\sqrt{\eta}\omega_{ci}} \sin(\phi_1 - \sqrt{\eta}\omega_{ci}t - \theta(t))] dk_x dk_y dk_z
\end{aligned} \tag{17}$$

where

$$\hat{k}_\perp^2(t) = (k_x - V_0't k_y)^2 + \frac{1}{\eta} k_y^2, \tag{18}$$

and  $\tan \theta = k_y/\sqrt{\eta}(k_x - V_0't k_y)$ . It follows from Eq.(17) that finite Larmor radius effect of the interaction of the perturbation with time independent wave numbers  $k_x, k_y, k_z$  with ion, Larmor orbit of which is observed in sheared coordinates as a spiral continuously stretched with time, appears identical analytically to the interaction of the perturbation with wave numbers  $k_x - V_0't k_y, k_y/\sqrt{\eta}, k_z$  with ion, which rotates on the elliptical orbit that is observed in the laboratory frame. The time dependence of the finite Larmor radius effect is the basic linear mechanism of the action of the velocity shear on waves and instabilities in plasma shear flow.

In what follows, we consider the equilibrium distribution function  $F_{i0}$  as a Maxwellian,

$$F_0 = \frac{n_0(X)}{(2\pi v_T^2)^{3/2}} \exp \left( -\frac{v_\perp^2 + v_z^2}{v_T^2} \right), \tag{19}$$

assuming the inhomogeneity of the density and temperature of plasma shear flow species on coordinate  $X$ . In this paper we assume, that velocity shearing rate  $V_0'$  is much less than the ion cyclotron frequency  $\omega_{ci}$ , and put  $\eta = 1$ . The solution of the Vlasov equation for the perturbation  $f(t, k_x, k_y, k_z, v_\perp, \phi, v_z, z_1)$  of the distribution function  $F$ ,  $f = F - F_0$  with known  $F_0$  is

$$f = \frac{e}{m} \int_{t_0}^t \left[ \frac{1}{\omega_c} \frac{\partial \varphi}{\partial Y} \frac{\partial F_0}{\partial X} - \frac{\omega_c}{v_\perp} \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F_0}{\partial v_\perp} + \frac{\partial \varphi}{\partial z_1} \frac{\partial F_0}{\partial v_z} \right] dt'. \tag{20}$$



Using solution (20) for all plasma species in Poisson equation for the potential  $\varphi(\mathbf{r}, t)$ ,

$$\Delta \varphi(\mathbf{r}, t) = -4\pi \sum_{\alpha=i,e} e_{\alpha} \int f_{\alpha}(\mathbf{v}, \mathbf{r}, t) d\mathbf{v}_{\alpha}, \quad (21)$$

we obtain integral equation<sup>13,14</sup>, which governs the temporal evolution of the separate spatial Fourier harmonic of the electrostatic potential  $\varphi(\mathbf{k}, t)$  in plasma shear flow and is capable of handling linear as well as nonlinear<sup>13</sup> evolution of the electrostatic instabilities of plasma shear flows.

### III. HYDRODYNAMIC ION TEMPERATURE GRADIENT INSTABILITY

In this section, we consider the temporal evolution of the hydrodynamic ion temperature gradient instability in plasma shear flow. By using the methodology stated in Sec.II, we obtain the integral equation for the electrostatic potential  $\Phi(\mathbf{k}, t) = \varphi(\mathbf{k}, t) \Theta(t - t_0)$ , where  $\Theta(t - t_0)$  is the unit-step Heaviside function (it is equal to zero for  $t < t_0$  and equal to unity for  $t \geq t_0$ )

$$\begin{aligned} & \int_{t_0}^t dt_1 \frac{d}{dt_1} \{ \Phi(\mathbf{k}, t_1) [-(1+T) + A_{0i}(t, t_1)] \} \\ & - \int_{t_0}^t dt_1 \frac{d}{dt_1} \left\{ \Phi(\mathbf{k}, t_1) A_{0i}(t, t_1) \left( 1 - e^{-\frac{1}{2}k_z^2 v_{Ti}^2 (t-t_1)^2} \right) \right\} \\ & + \int_{t_0}^t dt_1 \Phi(\mathbf{k}, t_1) A_{0i}(t, t_1) e^{-\frac{1}{2}k_z^2 v_{Ti}^2 (t-t_1)^2} \\ & \times \left( ik_y v_{di} - i\omega_{Ti} - k_z^2 v_{Ti}^2 (t-t_1) - \frac{i}{2} \omega_{Ti} k_z^2 v_{Ti}^2 (t-t_1)^2 \right) \\ & + i\omega_{Ti} \int_{t_0}^t dt_1 \Phi(\mathbf{k}, t_1) A_{1i}(t, t_1) e^{-\frac{1}{2}k_z^2 v_{Ti}^2 (t-t_1)^2} \\ & + T \int_{t_0}^t dt_1 \left( \frac{d\Phi(\mathbf{k}, t_1)}{dt_1} + ik_y v_{de} \Phi(\mathbf{k}, t_1) \right) e^{-\frac{1}{2}k_z^2 v_{Te}^2 (t-t_1)^2} = 0, \end{aligned} \quad (22)$$

where  $T = T_i/T_e$ ,  $\omega_{Ti} = k_y v_{di} d \ln T_i / d \ln n_i$ ,  $v_{di} = (cT_i/eB) d \ln n_{i0} / dx$  is the ion diamagnetic velocity,

$$A_{0i}(t, t_1) = I_0 \left( \hat{k}_{\perp}(t) \hat{k}_{\perp}(t_1) \rho_i^2 \right) e^{-\frac{1}{2}\rho_i^2 (\hat{k}_{\perp}^2(t) + \hat{k}_{\perp}^2(t_1))}, \quad (23)$$

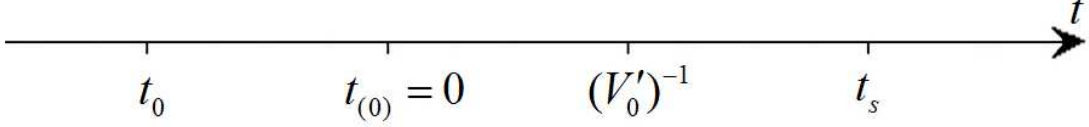


FIG. 1. The sequence of the characteristic times for Eq.(22) for the long wavelength perturbations with  $\hat{k}_\perp(t_0)\rho_i < 1$ .

$$\begin{aligned}
A_{1i}(t, t_1) &= e^{-\frac{1}{2}\rho_i^2(\hat{k}_\perp^2(t) + \hat{k}_\perp^2(t_1))} \\
&\times \left[ \left( 1 - \frac{\rho_i^2}{2} (\hat{k}_\perp^2(t) + \hat{k}_\perp^2(t_1)) \right) I_0(\hat{k}_\perp(t) \hat{k}_\perp(t_1) \rho_i^2) \right. \\
&\quad \left. + \rho_i^2 \hat{k}_\perp(t) \hat{k}_\perp(t_1) I_1(\hat{k}_\perp(t) \hat{k}_\perp(t_1) \rho_i^2) \right]. \tag{24}
\end{aligned}$$

Eq.(22) is the extension of the Eq.(25) in Ref.<sup>13</sup> onto the accounting for the inhomogeneity of the ion temperature. Naturally, it is not possible to obtain explicit single analytical presentation of the solution to Eq.(22), which is valid for any desired time. The exceptional advantage of the application of the non-modal approach, which uses the wavenumber-time variables, is a possibility to perform the analysis of the electrostatic potential evolution at any finite time domain and with an arbitrary initial time  $t_0$ . In our analysis, initiated in<sup>13</sup>, we distinguish different characteristic times of the evolution process of the long wavelength perturbations with  $\hat{k}_\perp(t_0)\rho_i \ll 1$ :  $t_0$  is the time of the perturbation origin;  $t_{(0)} = 0$  is the time of the electric field (shear flow) origin; times  $t_{(1)} = (V'_0)^{-1}$ ,  $t_{(2)} = t_s = (V'_0 k_y \rho_i)^{-1}$  define different stages of the non-modal evolution of the potential under the action of the shear flow. In these time intervals different tasks in the solving of Eq.(22) arise (as it was with Eq.(25) in Ref.<sup>13</sup>). We will derive the linear solution of Eq.(22) at time  $(V'_0)^{-1} \gg t \gg t_0$  of the modal instability development, at which shear flow effects are underdeveloped. Then, we will derive the solution at time  $t \gg (V'_0)^{-1}$ , at which the non-modal effects of the shear flow become pronounced. And, at last, we will derive solution at the final stage of the perturbation evolution at time  $t \gg t_s = (V'_0 k_y \rho_i)^{-1} \gg (V'_0)^{-1}$ , at which even initially long wavelength perturbations with  $\hat{k}_\perp \rho_i < 1$  at time  $t \ll t_s$ , becomes the short wavelength ones with  $\hat{k}_\perp \rho_i \gg 1$  at  $t \gg t_s$ .

The long-wavelength drift perturbations with  $\hat{k}_\perp \rho_i < 1$  at time  $t = 0$ , will be the long wavelength ones at time  $t < t_s = (V'_0 k_y \rho_i)^{-1}$ . For this limit, we use the approximations,

$$A_{0i}(t, t_1) \approx 1 - \frac{\rho_i^2}{2} (\hat{k}_\perp^2(t) + \hat{k}_\perp^2(t_1))$$

$$\approx b_i + \rho_i^2 \left( k_x k_y V'_0 (t + t_1) - \frac{1}{2} k_y^2 (V'_0)^2 (t^2 + t_1^2) \right) \Theta(t_1), \quad (25)$$

$$\begin{aligned} A_{1i}(t, t_1) &\approx 1 - \rho_i^2 \left( \hat{k}_\perp^2(t) + \hat{k}_\perp^2(t_1) \right) \\ &\approx b_{i1} + \rho_i^2 \left( 2k_x k_y V'_0 (t + t_1) - k_y^2 (V'_0)^2 (t^2 + t_1^2) \right) \Theta(t_1), \end{aligned} \quad (26)$$

where  $b_i = 1 - k_\perp^2 \rho_i^2$ ,  $b_{i1} = 1 - 2k_\perp^2 \rho_i^2$ ,  $k_\perp^2 = k_x^2 + k_y^2$ , and  $\Theta(t)$  indicates that the shear flow emerges at  $t = 0$ . In Eq.(22), we use the approximations  $1 - \exp(-(1/2)k_z^2 v_{Ti}^2 (t - t_1)^2) \simeq -(1/2)k_z^2 v_{Ti}^2 (t - t_1)^2$ , that corresponds to the weak ion Landau damping. Assuming that the magnitude of the initial potential  $\Phi(\mathbf{k}, t = t_0)$  is negligibly small, the integral

$$\begin{aligned} &\int_{t_0}^t dt_1 \frac{d}{dt_1} \left( \Phi(\mathbf{k}, t_1) \left( 1 - e^{-\frac{1}{2} k_z^2 v_{Ti}^2 (t - t_1)^2} \right) \right) \\ &\simeq \int_{t_0}^t dt_1 \frac{d}{dt_1} \left( \Phi(\mathbf{k}, t_1) \left( \frac{1}{2} k_z^2 v_{Ti}^2 (t - t_1)^2 \right) \right) \\ &= \frac{1}{2} k_z^2 v_{Ti}^2 (t - t_0)^2 \Phi(\mathbf{k}, t_0) \end{aligned} \quad (27)$$

may be ignored in Eq.(22). This approximation is reasonable only for the exponentially growing, faster than  $(t - t_0)$  or  $(t - t_0)^2$ , potential  $\Phi(\mathbf{k}, t)$ ; then  $\Phi(\mathbf{k}, t_0)(t - t_0)$  and  $(d\Phi(\mathbf{k}, t)/dt)|_{t=t_0} \cdot (t - t_0)^2$  are exponentially small with respect to  $\Phi(\mathbf{k}, t)$ . We assume that  $k_z^2 v_{Te}^2 (t - t_1)^2 \gg 1$ , that corresponds to adiabatic electrons, and neglect exponentially small electron terms in Eq.(22). In result, we obtain the simplified integral equation, which describes the linear evolution of the electrostatic potential at time  $t \ll t_s$ ,

$$\begin{aligned} &(T + k_\perp^2 \rho_i^2) \int_{t_0}^t dt_1 \frac{d^3 \Psi(\mathbf{k}, t_1)}{dt_1^3} \\ &- \int_{t_0}^t dt_1 \frac{d^2 \Psi(\mathbf{k}, t_1)}{dt_1^2} \left[ b_i (i k_y v_{di} - k_z^2 v_{Ti}^2 (t - t_1) \right. \\ &\quad \left. - \frac{i}{2} \omega_{Ti} k_z^2 v_{Ti}^2 (t - t_1)^2) - i \omega_{Ti} k_\perp^2 \rho_i^2 \right] \\ &= \rho_i^2 \int_{t_0}^t dt_1 \frac{d}{dt_1} \left[ \frac{d^2 \Psi(\mathbf{k}, t_1)}{dt_1^2} \left( k_x k_y V'_0 (t + t_1) - \frac{1}{2} k_y^2 (V'_0)^2 (t^2 + t_1^2) \right) \Theta(t_1) \right] \end{aligned}$$

$$\begin{aligned}
& + \rho_i^2 \int_{t_0}^t dt_1 \frac{d^2 \Psi(\mathbf{k}, t_1)}{dt_1^2} \left( k_x k_y V_0'(t + t_1) - \frac{1}{2} k_y^2 (V_0')^2 (t^2 + t_1^2) \right) \Theta(t_1) \\
& \times \left[ i k_y v_{di} + i \omega_{Ti} - k_z^2 v_{Ti}^2 (t - t_1) - i \frac{\omega_{Ti}}{2} k_z^2 v_{Ti}^2 (t - t_1)^2 \right]. \tag{28}
\end{aligned}$$

where new variable  $\Psi$ , determined as  $d^2 \Psi / dt^2 = \Phi$ , is introduced. Now we obtain the solution to Eq.(28) in the limit of large value of the parameter  $\eta_i \gg 1$ , at which long wavelength with  $k_{\perp} \rho_i < 1$  hydrodynamic ion temperature gradient instability is developed in a shearless plasma. By integration in parts of Eq.(28) and neglecting the initial values of  $d\Psi(\mathbf{k}, t)/dt$  and  $\Psi(\mathbf{k}, t)$  at  $t = t_0$  with assumption of the exponential growth for  $\Psi(\mathbf{k}, t)$  at time  $t \gg t_0$ , we obtain in the zero approximation the equation

$$\int_{t_0}^t dt_1 \left[ (T + k_{\perp}^2 \rho_i^2) \frac{d^3 \Psi(\mathbf{k}, t_1)}{dt_1^3} + i \omega_{Ti} k_z^2 v_{Ti} b_i \Psi(\mathbf{k}, t_1) \right] = 0, \tag{29}$$

in which the dominant terms in the left-hand side of Eq.(28) are retained, and the right-hand side, proportional to  $k_y^2 \rho_i^2 \ll 1$ , is neglected. The solution to Eq.(29) is  $\Psi(\mathbf{k}, t) = C \exp(-i\omega(\mathbf{k})t)$ , where the frequency  $\omega(\mathbf{k})$  is determined by the known equation

$$\omega^3(\mathbf{k}) = -\omega_{Ti} k_z^2 v_{Ti}^2 \frac{(1 - k_{\perp}^2 \rho_i^2)}{(T + k_{\perp}^2 \rho_i^2)}. \tag{30}$$

The accounting for the small right-hand side of Eq.(28) will modify that solution. We seek in the next approximation the solution for  $\Psi(\mathbf{k}, t)$  in the form

$$\Psi(\mathbf{k}, t) = C \exp(-i\omega(\mathbf{k})t + \sigma(\mathbf{k}, t)), \tag{31}$$

where the higher order correction,  $\sigma(\mathbf{k}, t)$ , we find by use the procedure of the successive approximations<sup>13</sup>, which gives the following equation for  $\sigma(\mathbf{k}, t)$ :

$$\begin{aligned}
& 3(T + k_{\perp}^2 \rho_i^2) \omega^2(\mathbf{k}) \int_{t_0}^t dt_1 \Psi(\mathbf{k}, t_1) \frac{d\sigma}{dt_1} \\
& = -\rho_i^2 \int_{t_0}^t dt_1 \frac{d}{dt_1} \left[ \frac{d^2 \Psi(\mathbf{k}, t_1)}{dt_1^2} \left( k_x k_y V_0'(t + t_1) - \frac{1}{2} k_y^2 (V_0')^2 (t^2 + t_1^2) \right) \Theta(t_1) \right] \\
& + i \frac{\omega_{Ti}}{2} \rho_i^2 \int_{t_0}^t dt_1 \frac{d^2 \Psi(\mathbf{k}, t_1)}{dt_1^2} \left( k_x k_y V_0'(t + t_1) - \frac{1}{2} k_y^2 (V_0')^2 (t^2 + t_1^2) \right) \Theta(t_1) \\
& \times k_z^2 v_{Ti}^2 (t - t_1)^2, \tag{32}
\end{aligned}$$

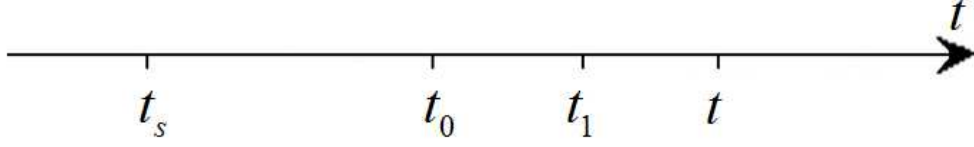


FIG. 2. The domain of the integration  $[t_0, t]$  over time in Eq.(36) for the perturbations with  $\hat{k}_\perp(t_0)\rho_i \gg 1$ .

where  $\sigma(\mathbf{k}, t)$  is neglected in  $\Psi(\mathbf{k}, t_1)$  in the right-hand side. The solution to Eq.(32) is obtained straightforwardly and is equal to

$$\sigma(\mathbf{k}, t) = -\frac{1}{3}k_y^2\rho_i^2\lambda(V_0't)^2 + \frac{i}{9}k_y^2\rho_i^2\omega(\mathbf{k})t\lambda(V_0't)^2, \quad (33)$$

where

$$\lambda = \frac{1 - T - 2k_\perp^2\rho_i^2}{(1 - k_\perp^2\rho_i^2)(T + k_\perp^2\rho_i^2)}.$$

This result displays, that linear effect of the shear flow at time  $t_s \gg t \gg (V_0')^{-1}$  consists in the non-modal decrease of the frequency and the growth rate with time. Qualitatively the same result was obtained for the drift kinetic instability in Ref.<sup>13</sup>.

It may be anticipated that the most substantial effect of the instability suppression will be attained at time  $t$  approaching  $t_s$  and that suppression will continue at  $t > t_s$ . For that time, however, solution to Eq.(22) may be obtained only numerically. For time  $t \gg t_s$ , we have  $\hat{k}_\perp(t)\rho_i \approx k_y V_0' t \rho_i = t/t_s \gg 1$  and small parameter  $t_s/t \sim t_s/t_0 \ll 1$  will appear in Eq.(22), that admits the receiving of the analytical solution<sup>13</sup> to Eq.(22). For that time

$$A_{0i}(t, t_1) \approx \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{1}{2t_s^2}(t-t_1)^2}, \quad (34)$$

$$A_{1i}(t, t_1) \approx \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{1}{2t_s^2}(t-t_1)^2} \left(1 - \frac{1}{2t_s^2}(t-t_1)^2\right). \quad (35)$$

For the application the asymptotics (34), (35) for time  $t_1$  in the whole interval  $t > t_1 > t_0$ , we have to consider in integral equation (32) the initial time  $t_0 > t_s$ . At that case the condition  $\varphi(\mathbf{k}, t) = 0$ , for  $(V_0')^{-1} < t_0$  which was used for the function  $\Phi(\mathbf{k}, t)$  in Eq.(22), in which time  $t_0$  was considered as the time preceding the development of the modal instability and appearance of the shear flow, is not applicable now, and we have to restore the equation for  $\varphi(\mathbf{k}, t)$ , obtained in Ref.<sup>13</sup>. It is important to note also, that for the decaying with time

solution for  $\varphi(\mathbf{k}, t)$ , the approximation of neglecting by initial value  $\varphi(\mathbf{k}, t_0)$  used in Eq.(28) is not justified and we have to retain these terms in equation considered. We obtain finally the following integral equation:

$$\begin{aligned}
(1+T) \int_{t_0}^t dt_1 \frac{d\varphi(\mathbf{k}, t)}{dt_t} &= \int_{t_0}^t dt_1 \frac{d}{dt_t} \left[ \varphi(\mathbf{k}, t) \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{\kappa_i^2}{2}(t-t_1)^2} \right] \\
&+ \int_{t_0}^t dt_1 \varphi(\mathbf{k}, t) \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{\kappa_i^2}{2}(t-t_1)^2} \\
&\times \left( i k_y v_{di} - \frac{i}{2} \omega_{Ti} \kappa_i^2 (t-t_1)^2 - k_z^2 v_{Ti}^2 (t-t_1) \right) \\
&+ \varphi(\mathbf{k}, t_0) \frac{t_s}{t_0 \sqrt{2\pi}} \left( e^{-\frac{1}{2t_s^2}(t-t_0)^2} - 1 \right), \tag{36}
\end{aligned}$$

where  $\kappa_i^2 = t_s^{-2} + k_z^2 v_{Ti}^2$ . At  $t \gg t_s$ , the right-hand side of Eq.(36) is proportional to small parameter  $t_s/t \sim t_s/t_0$ . In zero order in that parameter we have the equation  $\int_{t_0}^t dt_1 (d\varphi(\mathbf{k}, t_1)/dt_1) = 0$  with solution  $\varphi(\mathbf{k}, t) = \text{const}$ . The solution, which accounted for the terms of the order of  $t_s/t$  in Eq.(36), we find in the form  $\varphi(\mathbf{k}, t) = C \exp(\sigma(\mathbf{k}, t))$ , where  $\sigma(\mathbf{k}, t) = O(t_s/t)$ . Omitting  $\sigma(\mathbf{k}, t)$  in the right-hand part of Eq.(36), we obtain the approximate solution for  $\varphi(\mathbf{k}, t)$ ,

$$\varphi(\mathbf{k}, t) = \varphi_0 \exp \left[ \frac{t_s}{t(1+T)} \left( i \frac{k_y v_{di}}{2\kappa_i} - i \frac{\omega_{Ti}}{4\kappa_i} + \sqrt{\frac{2}{\pi}} \frac{k_z^2 v_{Ti}^2}{\kappa_i^2} \right) \right], \tag{37}$$

which is qualitatively the same as it was obtained in Ref.<sup>13</sup> for the kinetic drift instability for large time,  $t \gg t_s$  and is far from ordinary modal solution.

#### IV. NON-MODAL KINETIC EVOLUTION OF THE HYDRODYNAMIC RESISTIVE DRIFT INSTABILITY

In this section, we consider the temporal evolution of the resistive drift instability in plasma with comparable temperatures of ions and electrons. In convective electron guiding center coordinates, linear perturbation  $f_e$  of the steady state electron distribution function  $F_{0e}$  is governed by linearized Vlasov equation with number-conserving Bhatnagar-Gross-Krook (BGK) collision term<sup>17</sup>

$$\frac{\partial f_e}{\partial t} = \frac{e}{m_e \omega_{ce}} \frac{\partial \varphi}{\partial Y_e} \frac{\partial F_{0e}}{\partial X_e} - \frac{e}{m_e} \frac{\omega_{ce}}{v_\perp} \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F_{0e}}{\partial v_\perp} + \frac{e}{m_e} \frac{\partial \varphi}{\partial z} \frac{\partial F_{0e}}{\partial v_z} - \nu_e \left( f_e - \frac{n_{e1}(\mathbf{r}, t)}{n_{e0}} F_{0e} \right). \tag{38}$$

In Eq.(38)  $n_{e1}$  is the perturbed electron density, and  $X_e$  is electron leading center coordinate. The solution to Eq.(38) is

$$f_e = \frac{e}{m_e} \int_{t_0}^t e^{-\nu_e(t-t_1)} \left[ \frac{1}{\omega_{ce}} \frac{\partial \varphi}{\partial Y_e} \frac{\partial F_{e0}}{\partial X_e} - \frac{\omega_{ce}}{v_\perp} \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F_{e0}}{\partial v_\perp} + \frac{\partial \varphi}{\partial z_1} \frac{\partial F_{e0}}{\partial v_z} \right] dt_1 \\ + \nu_e \int_{t_0}^t e^{-\nu_e(t-t_1)} \frac{n_{e1}(\mathbf{r}, t_1)}{n_{e0}} F_{0e} dt_1. \quad (39)$$

Using solution (39) with Maxwellian distribution  $F_{0e}$ , (19), and solution (20) for the perturbation of the ion distribution function in Poisson equation (21), we obtain in the quasineutrality approximation the following equation for the potential  $\Phi(\mathbf{k}, t) = \varphi(\mathbf{k}, t) \Theta(t - t_0)$  for low frequency,  $d\varphi/dt \ll \omega_{ci}\varphi$ , perturbations of drift type:

$$\int_{t_0}^t dt_1 \left\{ (1 + T) \frac{d}{dt_1} \Phi(\mathbf{k}, t_1) - \frac{d}{dt_1} (\Phi(\mathbf{k}, t_1) A_{0i}(t, t_1)) e^{-\frac{1}{2} k_z^2 v_{Ti}^2 (t-t_1)^2} \right\} \\ = i \int_{t_0}^t dt_1 \Phi(\mathbf{k}, t_1) k_y v_{di} A_{0i}(t, t_1) + T G_e(\mathbf{k}, t), \quad (40)$$

where  $T = T_i/T_e$  and  $\Theta(t - t_0)$  is the unit-step Heaviside function.  $G_e(\mathbf{k}, t)$  determines the nonadiabatic part of the electron density perturbation,

$$n_{e1}(\mathbf{k}, t) = -\frac{en_{e0}}{T_e} \Phi(\mathbf{k}, t) + \frac{en_{e0}}{T_e} G_e(\mathbf{k}, t). \quad (41)$$

and is equal to

$$G_e(\mathbf{k}, t) = \int_{t_0}^t dt_1 \left( \frac{d}{dt_1} (\Phi(\mathbf{k}, t_1) e^{-\nu_e(t-t_1)}) + i k_y v_{de} \Phi(\mathbf{k}, t_1) e^{-\nu_e(t-t_1)} \right) e^{-\frac{1}{2} k_z^2 v_{Te}^2 (t-t_1)^2} \\ + 4\pi\nu_e \lambda_{De}^2 \int_{t_0}^t dt_1 n_{e1}(\mathbf{k}, t_1) e^{-\nu_e(t-t_1) - \frac{1}{2} k_z^2 v_{Te}^2 (t-t_1)^2}. \quad (42)$$

Now we apply the methodology developed in Ref.<sup>13</sup> to calculate the approximate solution to system (40), (42) for long wavelength perturbations with  $\hat{k}_\perp(t) \rho_i < 1$  for times limited by the condition  $(V'_0)^{-1} < t < t_s$ . Making use the approximation (23) we present Eq.(40) in the form

$$\int_{t_0}^t dt_1 \left( \frac{d\Phi(\mathbf{k}, t_1)}{dt_1} + i\omega(\mathbf{k}) \Phi(\mathbf{k}, t_1) \right)$$

$$\begin{aligned}
&= -\frac{b_i}{T + k_{\perp}^2 \rho_i^2} \int_{t_0}^t dt_1 \left( \frac{d\Phi(\mathbf{k}, t_1)}{dt_1} + ik_y v_{di} \Phi(\mathbf{k}, t_1) \right) \left( 1 - \exp \left( -\frac{1}{2} k_z^2 v_{Ti}^2 (t - t_1)^2 \right) \right) \\
&\quad + \frac{1}{T + k_{\perp}^2 \rho_i^2} \int_0^t dt_1 \left( \frac{d\Phi(\mathbf{k}, t_1)}{dt_1} + ik_y v_{di} \Phi(\mathbf{k}, t_1) \right) \left( \frac{k_x (t + t_1)}{k_y V_0' t_s^2} - \frac{(t^2 + t_1^2)}{2t_s^2} \right) \\
&\quad + \frac{1}{T + k_{\perp}^2 \rho_i^2} \int_0^t dt_1 \Phi(\mathbf{k}, t_1) \frac{1}{V_0' t_s^2} \left( \frac{k_x}{k_y} - V_0' t \right) + \frac{T}{T + k_{\perp}^2 \rho_i^2} G_e(\mathbf{k}, t), \tag{43}
\end{aligned}$$

where  $\omega(\mathbf{k})$  is

$$\omega(\mathbf{k}) = \frac{b_i}{1 + k_{\perp}^2 \rho_s^2} k_y v_{de}, \tag{44}$$

$\rho_s$  is the ion thermal Larmor radius with electron temperature, and  $v_{de}$  is the electron diamagnetic velocity. The first term in the right-hand side of Eq.(43) determines the ion Landau damping; this term is the same as in plasma without shear flow. The next three terms originate from shear flow and determine the corrections to the frequency and growth rate of the ordinary hydrodynamic resistive drift instability, which are provided by shear flow. The right-hand side of Eq.(43) is small for  $(V_0')^{-1} < t < t_s$ , and because of the weakness of ion Landau damping. We equate the right-hand side of Eq.(43) to zero and obtain in the lowest order the modal solution,  $\Phi(\mathbf{k}, t) = \Phi_0 \exp(-i\omega(\mathbf{k})t)$ . Making use this solution in Eq.(42) to evaluate  $G_e(\mathbf{k}, t)$ , we obtain

$$G_e(\mathbf{k}, t) = -i \frac{\frac{(\omega(\mathbf{k}) - k_y v_{de})}{k_z v_{Te}} \sqrt{\frac{\pi}{2}} W\left(\frac{\omega(\mathbf{k}) + i\nu_e}{\sqrt{2} k_z v_{Te}}\right)}{1 - \frac{\nu_e}{k_z v_{Te}} \sqrt{\frac{\pi}{2}} W\left(\frac{\omega(\mathbf{k}) + i\nu_e}{\sqrt{2} k_z v_{Te}}\right)} \Phi(\mathbf{k}, t), \tag{45}$$

where  $W(z) = e^{-z^2} \left( 1 + (2i/\sqrt{\pi}) \int_0^z e^{t^2} dt \right)$  is plasma dispersion function. Accounting for the small right-hand side of Eq.(43), we seek in the next approximation the solution in the form

$$\Phi(\mathbf{k}, t) = \Phi_0 \exp(-i\omega(\mathbf{k})t + \sigma(\mathbf{k}, t)). \tag{46}$$

Assuming that all terms at the right-hand side of Eq.(41) are of the same order, by inserting Eq.(46) into Eq.(43) and neglecting derivative  $d\sigma(\mathbf{k}, t)/dt$  in the right-hand side of Eq.(43), the equation for  $\sigma(\mathbf{k}, t)$  is found

$$\int_{t_0 \rightarrow -\infty}^t dt_1 \Phi(\mathbf{k}, t_1) \left[ \frac{d\sigma(\mathbf{k}, t_1)}{dt_1} - i\delta\omega(\mathbf{k}) \right]$$



$$-\frac{\Theta(t_1)}{(T + k_{\perp}^2 \rho_i^2) t_s^2} \left( i\omega(\mathbf{k}) t_1^2 \frac{(1+T)}{b_i} - 2t_1 \left( 1 + \frac{T + k_{\perp}^2 \rho_i^2}{2b_i} \right) \right) \Big] = 0, \quad (47)$$

in which was assumed that  $t_0 \rightarrow -\infty$ . In this equation,  $\delta\omega(\mathbf{k}) = \text{Re}\delta\omega(\mathbf{k}) + i\gamma(\mathbf{k})$ , where with assumption that  $\nu_e \gg \omega(\mathbf{k})$  and  $\nu_e \gg k_z v_{Te}$ ,

$$\text{Re} \delta\omega(\mathbf{k}) = -\frac{\omega^3(\mathbf{k}) k_{\perp}^2 \rho_s^2}{(1 + k_{\perp}^2 \rho_s^2) k_z^2 v_{Te}^2}, \quad (48)$$

and

$$\gamma(\mathbf{k}) = \frac{\nu_e \omega^2(\mathbf{k}) k_{\perp}^2 \rho_s^2}{(1 + k_{\perp}^2 \rho_s^2) k_z^2 v_{Te}^2} \quad (49)$$

is well known growth rate of the resistive drift instability<sup>18</sup>. We obtain for  $(V_0')^{-1} < t < t_s$  the solution to Eq.(47) in the form

$$\begin{aligned} \Phi(\mathbf{k}, t) = \Phi_0 \exp \left[ -i\omega(\mathbf{k}) \left( t - \frac{(1+T)t^3}{3(T + k_{\perp}^2 \rho_i^2) t_s^2 b_i} \right) + i\text{Re} \delta\omega(\mathbf{k}) t \right. \\ \left. + \left( \gamma(\mathbf{k}) t - \frac{t^2}{(T + k_{\perp}^2 \rho_i^2) t_s^2} \left( 1 + \frac{T + k_{\perp}^2 \rho_i^2}{2b_i} \right) \right) \right], \end{aligned} \quad (50)$$

in which effect of the shear flow is the same as it was obtained in the linear non-modal theory of the kinetic drift instability<sup>13</sup> – shear flow leads to the decrease the frequency and the growth rate of the resistive drift instability with time.

In long-time limit,  $t \gg t_s$ , arguments of Bessel function  $I_0$  in Eq.(40) become large,  $\hat{k}(t) \hat{k}(t_1) \rho_i^2 \simeq t^2/t_s^2 \gg 1$  and asymptotics (34) simplifies Eq.(40) for potential  $\varphi(\mathbf{k}, t)$ ,

$$\begin{aligned} (1+T) \int_{t_0}^t dt_1 \frac{d\varphi(\mathbf{k}, t)}{dt_t} = \int_{t_0}^t dt_1 \frac{d}{dt_t} \left[ \varphi(\mathbf{k}, t) \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{\kappa_i^2}{2}(t-t_1)^2} \right] \\ + ik_y v_{di} \int_{t_0}^t dt_1 \varphi(\mathbf{k}, t) \frac{t_s}{\sqrt{2\pi t t_1}} e^{-\frac{\kappa_i^2}{2}(t-t_1)^2} \\ + \varphi(\mathbf{k}, t_0) \frac{t_s}{t_0 \sqrt{2\pi}} \left( e^{-\frac{1}{2t_s^2}(t-t_0)^2} - 1 \right), \end{aligned} \quad (51)$$

where  $\kappa_i^2 = t_s^{-2} + k_z^2 v_{Ti}^2$ . At  $t \gg t_s$ , the right hand side part of Eq.(51) is proportional to small parameter  $t_s/t \sim t_s/t_0$ . Applying the methodology of the solution of Eq.(36) to Eq.(51), we obtain the following solution:

$$\varphi(\mathbf{k}, t) = \varphi_0 \exp \left[ \frac{1}{(1+T)} \frac{t_s}{t} \left( i \frac{k_y v_{di}}{2\kappa_i} + \sqrt{\frac{2}{\pi}} \frac{1}{k_z^2 v_{Ti}^2 t_s^2} \right) \right], \quad (52)$$

which reproduces qualitatively the same time dependence as of solution (37).

## CONCLUSIONS

In this paper, we develop the non-modal kinetic theory of the hydrodynamic ion temperature gradient and resistive drift instabilities in plasma shear flow. In this theory, which is grounded on the shearing modes approach in the kinetic theory of plasma shear flows<sup>13</sup>, the shear flow reveals as the time-dependent effect of the finite Larmor radius in the integral equation for the electrostatic potential. This effect is of principal importance for turbulence evolution in plasma shear flows. It consists in the interaction of ions undergoing cyclotron motion with inhomogeneous electric field of sheared modes, which due to their distortion by shear flows have time dependent wave number in laboratory frame. In Ref.<sup>13</sup> we obtain, that this effect is a source of the enhanced suppression of the kinetic drift instability by shear flow. In this paper we find, that in spite of their hydrodynamic nature, ion temperature gradient and resistive drift instabilities in shear flow at times  $t > (V_0')^{-1}$  pass through the same linear non-modal kinetic processes of their evolution, as the kinetic drift instability. These processes reveal in the non-modal decrease with time the frequency and the growth rate of the unstable perturbations that display the universality in the linear description of the temporal evolution of the electrostatic drift instabilities of shear flows.

It is important to note, that being qualitatively similar in their linear evolutionary theory, these two discussed instabilities have different nonlinear descriptions. The renormalized nonmodal nonlinear theory<sup>13</sup>, which accounts for the effect of turbulent scattering of ions by the ensemble of the sheared modes with randomly distributed initial phases, developed for the kinetic drift instability, is completely applicable to the resistive drift instability and gives completely the same result, determined by equation (68) in Ref.<sup>13</sup>, where the frequency and the growth rate are determined now by Eqs.(48), (49). This nonlinear effect, which is absent in conventional gyrokinetic theory, consists in the scattering of the ion gyration angle by sheared perturbations and reveals in the reduction of the growth rate with time as  $\gamma (V_0't)^{-6}$ .

The hydrodynamic ion temperature gradient instability considered here have the modal growth rate of the order of its frequency. Therefore, the methodology of the renormalized nonlinear theory, developed in Ref.<sup>13</sup>, which require the smallness of the growth rate in comparison with frequency, is not applicable for that instability and new nonlinear theory have to be developed.

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## Appendix A: TRANSFORMATION TO SHEARED COORDINATES FOR NON-STATIONARY SHEAR FLOW

The procedure of the transformation of the Vlasov-Poisson system to the sheared coordinates, developed in Ref.<sup>13</sup>, is generalized easily on the case of the spatially homogeneous, but time dependent velocity shear,

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{V}_0(x, t) = -\frac{c}{B} \frac{\partial E_0(x, t)}{\partial x} x \mathbf{e}_y = \frac{\partial V_0(x, t)}{\partial x} x \mathbf{e}_y = V'_0 \frac{da(t)}{dt} x \mathbf{e}_y. \quad (\text{A1})$$

where  $V'_0$  is a parameter with dimension of the velocity shear and  $a(t)$  is a function with dimension of time. The transition in the Vlasov equation from coordinates  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  and velocity  $\hat{\mathbf{v}}$  to convected with flow velocity coordinates  $\hat{v}_x = v_x$ ,  $\hat{v}_y = v_y + V'_0(da/dt)x$ ,  $\hat{v}_z = v_z$ , and sheared coordinates, determined by  $\hat{x} = x$ ,  $\hat{y} = y + V'_0 a(t)x$ ,  $\hat{z} = z$  (it is assumed that inhomogeneous electric field, and respectively shear flow originate at time  $t = 0$  and time ordering  $\omega_{ci}T \gg 1$  is adapted, where  $T$  is characteristic time of the flow velocity variations) removes from the Vlasov equation spatial inhomogeneity introduced by shear flow. With leading center coordinates  $X$ ,  $Y$ , determined in convective set of reference,

$$X = x + \frac{v_\perp}{\sqrt{\eta(t)}\omega_c} \sin \phi, \quad (\text{A2})$$

$$Y = y - \frac{v_\perp}{\eta(t)\omega_c} \cos \phi - V'_0 a(t) (X - x), \quad z_1 = z - v_z t, \quad (\text{A3})$$

and velocity space coordinates

$$v_x = v_\perp \cos \phi, \quad v_y = \sqrt{\eta} v_\perp \sin \phi, \quad \phi = \phi_1 - \omega_c \mu(t), \quad v_z = v_z, \quad (\text{A4})$$

where

$$\eta(t) = 1 + \frac{V'_0}{\omega_c} \frac{da(t)}{dt}, \quad \mu(t) = \int_0^t \sqrt{\eta(t_1)} dt_1, \quad (\text{A5})$$

we obtain Vlasov equation in form (16) with time dependent coefficient  $\eta$ . The electrostatic potential  $\varphi(x, y, z, t)$  in Eq.(16) is determined by (17), where now

$$\hat{k}_\perp^2(t) = (k_x - V'_0 a(t) k_y)^2 + \frac{1}{\eta(t)} k_y^2, \quad (\text{A6})$$

and  $\tan \theta = k_y / \sqrt{\eta(t)} (k_x - V'_0 a(t) k_y)$ . Therefore, the results for the time dependent shear (A1) are reproduced easily from ones obtained for the stationary shear by the replacements  $V'_0 t$  on  $V'_0 a(t)$ . For real experimental conditions  $V'_0 \ll \omega_{ci}$ ; so  $\eta = 1$  and  $\mu = 1$  and in such case Eq.(16) still has not explicit time dependence. As a sample of such analysis, we present here the extension of the solution Eq.(50), obtained with assumption of  $\eta = 1$ , on the time dependent velocity shear

$$\begin{aligned} \Phi(\mathbf{k}, t) = \Phi_0 \exp & \left[ -i\omega(\mathbf{k}) \left( t - \frac{1+T}{(T+k_\perp^2 \rho_i^2) t_s^2 b_i} \int_0^t a^2(t_1) dt_1 \right) + i \text{Re } \delta\omega(\mathbf{k}) t \right. \\ & \left. + \left( \gamma(\mathbf{k}) t - \frac{a^2(t)}{(T+k_\perp^2 \rho_i^2) t_s^2} \left( 1 + \frac{T+k_\perp^2 \rho_i^2}{2b_i} \right) \right) \right]. \end{aligned} \quad (\text{A7})$$

The performed analysis displays, that the effects of shear flows becomes appreciable when the time-dependent part of the wave number (A6), becomes dominant, i.e. when  $|V'_0 a(t)| \gg 1$ . That condition may occur for the steady or for the growing with time velocity shear. For the velocity shear (A1) oscillating with time as  $da(t)/dt \sim \sin \omega_0 t$ , we have  $|V'_0 a(t)| \lesssim V'_0 / \omega_0$  and condition  $|V'_0 a(t)| \gg 1$  may be attained only for low frequency oscillation of the velocity shear, when the condition  $|V'_0| \gg \omega_0$  is met; effect of shear flow is negligible for  $\omega_0 > |V'_0|$ .

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